TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 290, Number 2, August 1985

DIRECTED SETS AND COFINAL TYPES

BY

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ABSTRACT. We show that 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$ are the only cofinal types of directed sets of size $\aleph_1$, but that there exist many cofinal types of directed sets of size continuum.

A partially ordered set D is directed if every two elements of D have an upper bound in D. In this note we consider some basic problems concerning directed sets which have their origin in the theory of Moore-Smith convergence in topology [**12**, **3**, **19**, **9**]. One such problem is to determine "all essential kind of directed sets" needed for defining the closure operator in a given class of spaces [**3**, p. 47]. Concerning this problem, the following important notion was introduced by J. Tukey [**19**]. Two directed sets D and E are *cofinally similar* if there is a partially ordered set C in which both can be embedded as cofinal subsets. He showed that this is an equivalence relation and that D and E are cofinally similar iff there is a convergent map from D into E and also a convergent map from E into D. The equivalence classes of this relation are called *cofinal types*. This concept has been extensively studied since then by various authors [**4**, **13**, **7**, **8**]. Already, from the first introduction of this concept, it has been known that 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$ represent different cofinal types of directed sets of size $\leqslant \aleph_1$, but no more than five such types were known. The main result of this paper shows that 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$ are the only cofinal types of convergence in spaces of character $\leqslant \aleph_1$ which can be constructed without additional set-theoretic assumptions. On the other hand, we shall construct many different cofinal types of directed sets of size continuum. This gives a solution to Problem 1 of J. Isbell [**7**]. The paper also contains several results about the structure of the class of all cofinal types, as well as a result about decomposing arbitrary partially ordered sets into directed sets. The results of this note were proved in February-March 1982 and presented to the ASL in January 1983.

1. A decomposition theorem. In this section we show that an arbitrary partially ordered set can be decomposed into a number of its directed subsets depending on the sizes of its antichains. This result is connected with an unpublished problem of F. Galvin concerning the Dilworth decomposition theorem [**6**] and it generalizes a similar result of E. Milner and K. Prikry [**11**]. The transitivity condition of a partial ordering is not used in our proof, so we state our result so as to apply to an arbitrary

Received by the editors December 3, 1984.
1980 *Mathematics Subject Classification*. Primary 03E05, 03E35; Secondary 06A10, 18B35, 54A15.

binary relation rather than to a partial ordering itself. In this form the result also subsumes a well-known partition relation for cardinals and it might have some further applications.

Let $\langle A, R\rangle$ be a fixed structure, where R is a binary relation on A, and let κ, λ be cardinals such that $\kappa \geqslant \omega$ and $\lambda \geqslant 3$. We say that $D \subseteq A$ is a *λ-directed subset* of $\langle A, R\rangle$ iff for all $C \in [D]^{<\lambda}$ there is a $d \in D$ such that for all $c \in C$, $c\, R\, d$ or $c = d$. If $a, b \in A$, then $a \parallel_R b$ denotes the fact that $a \neq b$ and $(a, b) \notin R$ and $(b, a) \notin R$. We say that $B \subseteq A$ is an antichain of $\langle A, R\rangle$ iff $a \parallel_R b$ for all $a, b \in B$, $a \neq b$. By $\kappa \ll \lambda$ we denote the fact that $\kappa < \lambda$ and $\rho^\sigma < \lambda$ for all $\sigma < \kappa$ and $\rho < \lambda$.

THEOREM 1. *Assume λ is regular and $\kappa \ll \lambda$. Then any structure $\langle A, R\rangle$ with no antichain of size κ is the union of $< \lambda$ of its λ-directed subsets.*

PROOF. Let θ be a large enough regular cardinal and let $M \prec H_\theta$ be such that κ, λ, $\langle A, R\rangle \in M$, $M \cap \lambda \in \mathrm{Ord}$, $[M]^{<\kappa} \subseteq M$ and M has size $< \lambda$. We claim that

$$A = \bigcup\{D\colon D \in M \text{ and } D \text{ is a } \lambda\text{-directed subset of } \langle A, R\rangle\}.$$

Otherwise, pick a $b \in A$ not in this union. By induction on $\alpha < \kappa$ we define an antichain $\{b_\alpha\colon \alpha < \kappa\} \subseteq A \cap M$ as follows. Assume $\alpha < \kappa$ and $\{b_\beta\colon \beta < \alpha\} \subseteq A \cap M$ is an antichain of $\langle A, R\rangle$ such that $b_\beta \parallel b$ for all $\beta < \alpha$. Define $D = \{d \in A\colon b_\beta \parallel d \text{ for all } \beta < \alpha\}$. Then by our assumptions on M, $D \in M$ and $b \in D$. Hence D is not λ-directed. So there is a $C \in [D]^{<\lambda}$ in M which has no upper bounds in D. So in particular b is not an upper bound of C. Since $C \subseteq M$ there must be a $c \in C$ such that $c \parallel b$. So we can put $b_\alpha = c$. This completes the proof.

THEOREM 2. *Every structure $\langle A, R\rangle$ with no antichain of size κ is the union of $\leqslant \lambda^\kappa$ of its λ^+-directed subsets.*

PROOF. If κ is a regular cardinal the result follows from Theorem 1. So assume $\kappa > \mathrm{cf}\,\kappa$ and fix $\langle \kappa_\xi\colon \xi < \mathrm{cf}\,\kappa\rangle \uparrow \kappa$. Pick a chain $\langle M_\xi\colon \xi < \mathrm{cf}\,\kappa\rangle$ of elementary submodels of H_θ such that $\kappa, \lambda, \langle A, R\rangle \in M_\xi$, $M_\xi \cap \lambda^+ \in \mathrm{Ord}$, $[M_\xi]^{\kappa_\xi} \subseteq M_\xi$, M_ξ has size λ^{κ_ξ} and $M_\xi \in M_{\xi+1}$. Working as in the proof of Theorem 1 one shows that

$$A = \bigcup\left\{D\colon D \in \bigcup_{\xi < \mathrm{cf}\,\kappa} M_\xi \text{ and } D \text{ is } \lambda^+\text{-directed}\right\}.$$

This will clearly finish the proof of Theorem 2.

Assume λ is regular and $\kappa \ll \lambda$. Let $[\lambda]^2 = K_0 \cup K_1$ be a given partition. Define $R \subseteq \lambda \times \lambda$ by $(\alpha, \beta) \in R$ iff $\alpha < \beta$ and $\{\alpha, \beta\} \in K_0$. If there is no $B \in [\lambda]^\kappa$ such that $[B]^2 \subseteq K_1$, then by Theorem 1 there must be a $C \in [\lambda]^\lambda$ such that $[C]^2 \subseteq K_0$. Hence the well-known partition relation $\lambda \to (\lambda, \kappa)^2$ is a consequence of Theorem 1.

Assume κ and λ are as above and D is a directed set of cofinality $\mathrm{cf}\, D$ equal to λ. Applying Theorem 1 to D it follows that D contains a cofinal subset which is the union of $< \lambda$ chains of order type λ. This has been proved first by R. Laver [**10**, p. 100] for the case $\kappa = \aleph_0$ and then by Milner and Prikry [**11**] in general.

Note that the proof of Theorem 1 also gives the following fact: If κ is a weakly compact cardinal, then any structure $\langle A, R\rangle$ with no antichain of size κ is the union of $< \kappa$ of its κ-directed subsets.

2. Cofinal types. We begin this section with a central notion in the theory of directed sets introduced by Tukey [**19**]. For two directed sets D and E, we say that D is *cofinally finer* than E, and write $D \geqslant E$, iff there exists a convergent map from D into E, i.e., a function $f: D \to E$ such that for all $e \in E$ there is a $d \in D$ such that $f(c) \geqslant e$ for all $c \geqslant d$. It is clear that $\geqslant$ is a transitive relation on the class of all directed sets. The following useful fact is implicit in Tukey [**19**] and explicit in Schmidt [**13**].

PROPOSITION 1 [**19, 13**]. *$D \geqslant E$ iff there is a map $g: E \to D$ which maps unbounded sets into unbounded sets.*

PROOF. Let $f: D \to E$ be convergent. Pick a function $g: E \to D$ such that $f(c) \geqslant e$ for all $c \geqslant g(e)$ in D. Then g maps unbounded sets into unbounded sets.

Conversely, let $g: E \to D$ map unbounded sets into unbounded sets. Pick an f: $D \to E$ such that for all $d \in D, f(d)$ is an upper bound of $\{e \in E: g(e) \leqslant d\}$. Then f converges.

A function $g: E \to D$ which maps unbounded sets into unbounded sets is called a *Tukey function*. The next result of Tukey [**19**] shows that the relation of cofinal similarity is an equivalence relation and that in fact it coincides with the relation $\equiv$ generated by the quasi-ordering $\geqslant$.

THEOREM 3 [**19**]. *$D \geqslant E$ and $E \geqslant D$ iff there is a partially ordered set C in which both D and E can be embedded as cofinal subsets.*

PROOF. Only the direct implication is nontrivial. So assume $D \geqslant E$ and $E \geqslant D$. Working as in the proof of Proposition 1, we can find functions $f: D \to E$ and g: $E \to D$ such that if $d \in D$ and $e \in E$, then $d \geqslant_D g(e)$ implies $f(d) \geqslant_E e$, and $e \geqslant_E f(d)$ implies $g(e) \geqslant_D d$. Assume $D \cap E = \varnothing$ and define $\leqslant_*$ on $D \cup E$ as follows: $\leqslant_*$ on D is equal to $\leqslant_D$ and $\leqslant_*$ on E is equal to $\leqslant_E$. For $d \in D$ and $e \in E$ we put $e \leqslant_* d$ iff $g(e') \leqslant_D d$ for some $e' \geqslant_E e$, and $d \leqslant_* e$ iff $f(d') \leqslant_E e$ for some $d' \geqslant_D d$. Then $\leqslant_*$ is a quasi-ordering on $D \cup E$. Let $\sim$ be the equivalence relation generated by $\leqslant_*$. Then D and E are isomorphic to cofinal subsets of $\langle D \cup E/\sim, \leqslant_*\rangle$ via mappings $d \to [d]$ and $e \to [e]$, respectively.

The next simple and useful result of Tukey [**19**] shows that the class of all cofinal types forms an upper semilattice.

PROPOSITION 2 [**19**]. *If n is finite, then $D_1 \times \cdots \times D_n$ is the least upper bound of $D_1, \ldots, D_n$.*

PROOF. Suppose $f_i: E \to D_i$ converges for $i = 1, \ldots, n$. Define $f: E \to D_1 \times \cdots \times D_n$ by $f(e) = \langle f_1(e), \ldots, f_n(e)\rangle$. Then f is convergent.

In [**7**], Isbell showed that the upper semilattice of cofinal types is not a lattice and that Proposition 2 does not hold for infinite n.

Standard examples of directed sets are sets of the form $[\kappa]^{<\lambda}$ ordered by $\subseteq$. Note that by Proposition 1, $[\kappa]^{<\omega}$ is cofinally finer than any directed set of size $\leqslant \kappa$. The particular place of sets of the form $[\kappa]^{<\lambda}$ in the class of all directed sets has been studied by various authors [**4, 13, 7**]. Many natural questions about directed sets of the form $[\kappa]^{<\lambda}$ have also been considered by various authors in connection with the well-known singular cardinals problem in set theory, and many of them are still open. For example, it can be proved that $[\omega_\omega]^{\aleph_0} \equiv [\omega_{\omega+1}]^{\aleph_0}$ holds iff there is a family $\mathscr{F} \subseteq \mathscr{P}(\omega_\omega)$ of size $> \aleph_\omega$ such that $\{f \cap X: f \in \mathscr{F}\}$ has size $\leqslant$ size of X for all infinite $X \subseteq \omega_\omega$ of size $< \aleph_\omega$, which is a well-known set-theoretical statement. We shall now give a natural generalization of sets of the form $[\kappa]^{<\lambda}$ in order to produce many nonequivalent directed sets. We shall restrict ourselves to the case $\lambda = \aleph_1$ since this suffices for all of our applications, but let us note that all our definitions and results have obvious generalizations to higher cardinals λ. By $\mathscr{D}_\kappa$ we shall denote the set of all cofinal types of directed sets of size $\leqslant \kappa$. Note that $\mathscr{D}_{\aleph_0} = \{1, \omega\}$, so the first nontrivial problem is to determine the possible structure of $\mathscr{D}_{\aleph_1}$.

Let $\lim(\omega)$ be the class of all ordinals of cofinality $\leqslant \omega$, and let $S \subseteq \lim(\omega)$ be a nonempty set. Then by $D(S)$ we denote the set of all countable sets $C \subseteq S$ such that $\sup(C \cap \alpha) \in C$ for all α. We consider $D(S)$ as a directed set partially ordered by $\subseteq$. It is interesting to note that many known examples of cofinal types can be represented with $D(S)$ for some $S \subseteq \lim(\omega)$. For example,

$$D(1) \equiv 1, \qquad D(\omega) \equiv \omega, \qquad D(\omega_1) \equiv \omega_1,$$

$$D(\omega_1 \setminus \{\omega\}) \equiv \omega \times \omega_1, \qquad D(\{\alpha + 1: \alpha < \omega_1\}) \equiv [\omega_1]^{<\omega}, \quad \text{etc.}$$

LEMMA 1. *Let $\kappa > \omega$ be regular and let S, $S' \subseteq \lim(\omega) \cap \kappa$ be such that S' is unbounded in κ and $\lim(\omega) \setminus S'$ is stationary in κ. Then $D(S) \geqslant D(S')$ implies that $S \setminus S'$ is nonstationary in κ and that S is unbounded in κ.*

PROOF. Let $g: D(S') \to D(S)$ be a given function and let us assume that $S \setminus S'$ is stationary in κ. We shall show that g is not a Tukey function. A very similar proof will show that g is not Tukey if S is bounded in κ.

Let θ be a large enough regular cardinal. Pick a countable $N \prec H_\theta$ such that g, S, $S' \in N$ and $\sup(N \cap \kappa) \in S \setminus S'$. Let F be the closure of $N \cap \kappa$, and let $\{\alpha_n: n < \omega\}$ be an enumeration of $F \setminus S$.

Since $N \prec H_\theta$ by induction on $n < \omega$ we can choose a sequence $S'_0 \supseteq S'_1 \supseteq \cdots$ of members of $N \cap [S']^\kappa$ and a sequence $\langle I_n: n < \omega \rangle$ of intervals of κ such that

(a) $\alpha_n \in I_n$ and $I_n \in N$,

(b) $g(\{\delta\}) \cap I_n = \varnothing$ for all $\delta \in S'_n$.

Now pick increasing $\delta_n \in N \cap S'_n$ $(n < \omega)$ converging to $\sup(N \cap \kappa)$ and let

$$C = \overline{\bigcup_{n<\omega} g(\{\delta_n\})}.$$

Then $C \subseteq S$ and C is an upper bound of $\{g(\{\delta_n\}): n < \omega\}$ in $D(S)$. But $\{\{\delta_n\}: n < \omega\}$ has no upper bounds in $D(S')$, so g is not Tukey.

THEOREM 4. Card $\mathscr{D}_{\kappa^{\aleph_0}} \geqslant 2^\kappa$ *for all regular* κ.

PROOF. Let $\mathscr{S}$ be a family of size 2^κ of subsets of $\lim(\omega) \cap \kappa$ such that $S \setminus S' \in \operatorname{stat} \kappa$ for all distinct S, $S' \in \mathscr{S}$. Then by Lemma 1, $D(S) \not\geqslant D(S')$ for all S, $S' \in \mathscr{S}$ with $S \neq S'$.

COROLLARY 5 (GCH). Card $\mathscr{D}_\kappa = 2^\kappa$ *for all regular* $\kappa > \omega$.

Let us now consider the Cartesian product of two directed sets of the form $D(S)$.

LEMMA 2. *Let $\kappa > \omega$ be regular and let $S, S' \subseteq \lim(\omega) \cap \kappa$ be unbounded in κ. Then $D(S) \times D(S') \geqslant [\kappa]^{<\omega}$ iff $S \cap S'$ is nonstationary in κ.*

PROOF. Let $S \cap S'$ be stationary and let $g: [\kappa]^{<\omega} \to D(S) \times D(S')$ be given. Pick a countable $N \prec H_\theta$ such that $g, S, S' \in N$ and $\sup(N \cap \kappa) \in S \cap S'$. Working as in the proof of Lemma 1, we can find an increasing sequence $\{\delta_n: n < \omega\} \subseteq N \cap \kappa$ such that

$$\overline{\bigcup_{n<\omega} \pi_1 \circ g(\{\delta_n\})} \subset S \quad \text{and} \quad \overline{\bigcup_{n<\omega} \pi_2 \circ g(\{\delta_n\})} \subseteq S',$$

whence g is not a Tukey function.

Conversely, assume $S \cap S'$ is nonstationary in κ. Pick a club C such that $C \cap S \cap S' = \varnothing$. Pick sequences $\langle \alpha_\xi: \xi < \kappa\rangle$ and $\langle \alpha'_\xi: \xi < \kappa\rangle$ in S and S', respectively, such that for all $\xi < \eta < \kappa$ there is a $\gamma \in C$ such that $\alpha_\xi, \alpha'_\xi < \gamma < \alpha_\eta, \alpha'_\eta$. Let $X = \{\langle\{\alpha_\xi\}, \{\alpha'_\xi\}\rangle: \xi < \kappa\}$. Then X is a subset of $D(S) \times D(S')$ of size κ, every infinite subset of which is unbounded in $D(S) \times D(S')$. Thus $D(S) \times D(S') \geqslant [\kappa]^{<\omega}$.

Since $D(S) \geqslant [\kappa]^{<\omega}$ iff $D(S) \times D(\lim(\omega) \cap \kappa) \geqslant [\kappa]^{<\omega}$, Lemma 2 gives the following

LEMMA 3. *Let $\kappa > \omega$ be regular and let $S \subseteq \lim(\omega) \cap \kappa$ be unbounded in κ. Then $D(S) \geqslant [\kappa]^{<\omega}$ iff S is nonstationary in κ.*

THEOREM 6. *For every regular cardinal $\kappa > \omega$ there exist directed sets D and E of size $\kappa^{\aleph_0}$ such that $D, E \ngeqslant [\kappa]^{<\omega}$ but $D \times E \geqslant [\kappa]^{<\omega}$.*

The best previous result in the direction of Theorem 4 is due to Isbell [**8**] who showed that Card $\mathscr{D}_{2^{\aleph_0}} \geqslant 7$. Concerning Corollary 5, let us note that J. Steprāns [**16**] (see also [**5**]) has previously shown, using a forcing argument, that Card $\mathscr{D}_{\aleph_1} = 2^{\aleph_1}$ is consistent. In his model CH holds. Theorem 6 solves Problem 3 of Isbell [7]. Let us note that, in L, Theorems 4–6 also hold for all singular cardinals κ.

3. Martin's Axiom and cofinal types. In the last two sections of this paper we consider the problem about the possible size of the set $\mathscr{D}_{\aleph_1}$ of all cofinal types of directed sets of cardinality $\leqslant \aleph_1$. The following simple fact is implicit in Tukey [**19**] and more explicit in Isbell [7, **8**].

PROPOSITION 3 [**19**, 7, **8**]. *Let D be a directed set of size $\leqslant \aleph_1$. Then either $D \equiv 1$, or $D \equiv \omega$, or $D \equiv \omega_1$, or $[\omega_1]^{<\omega} \geqslant D \geqslant \omega \times \omega_1$.*

PROOF. We may assume $\operatorname{cf} D = \aleph_1$ in which case $D \geqslant \omega_1$. If D contains no unbounded countable subset, then clearly $D \equiv \omega_1$. Otherwise $D \geqslant \omega$ so, by Proposition 2, $D \geqslant \omega \times \omega_1$.

Note also the following consequence of a result mentioned in §1. If D is a directed set of size $\leqslant \aleph_1$ with no infinite antichain, then either $D \equiv 1$, or $D \equiv \omega$, or $D \equiv \omega_1$, or $D \equiv \omega \times \omega_1$.

All relations between the basic five elements of $\mathscr{D}_{\aleph_1}$ are straightforward [**19**]: 1 is the minimal element and $[\omega_1]^{<\omega}$ is the maximal element of $\mathscr{D}_{\aleph_1}$; ω and ω_1 are two immediate (incomparable) successors of 1 with join $\omega \times \omega_1$. The next result shows the strong effect of Martin's Axiom on cofinal types of size $\aleph_1$.

THEOREM 7 ($\mathrm{MA}_{\aleph_1}$). *Let D be a directed set of size $\aleph_1$ in which every uncountable set contains a countable subset unbounded in D. Then $D \equiv [\omega_1]^{<\omega}$.*

PROOF. Clearly $\mathrm{cf}\, D = \aleph_1$ so we may and will assume that the domain of D is equal to ω_1 and that $\alpha <_D \beta$ implies $\alpha < \beta$. Let us consider the following two cases:

Case I. There is a strictly increasing continuous sequence $\langle N_\xi \colon \xi < \omega_1 \rangle$ of countable elementary submodels of $H_{\aleph_3}$ containing D such that the following set is cofinal in D.

$$E = \{ d \in D \colon \text{if } \xi < \omega_1 \text{ and if } d \notin N_{\xi+1} \text{ then there is } c \in N_{\xi+1} \setminus N_\xi \text{ such that } c \leqslant_D d \}.$$

Define $\mathscr{P}_1$ to be the set of all pairs $p = \langle A_p, B_p \rangle$, where A_p and B_p are finite subsets of E. We consider $\mathscr{P}_1$ as a partially ordered set ordered by $q \leqslant p$ iff $A_q \supseteq A_p$, $B_q \supseteq B_p$ and $a \nleqslant_D b$ for all $a \in A_q \setminus A_p$ and $b \in B_p$.

CLAIM 1. $\mathscr{P}_1$ is a ccc poset.

PROOF. Let $\langle p_\alpha \colon \alpha < \omega_1 \rangle$ be a given sequence of elements of $\mathscr{P}_1$. We may assume that for some $n \geqslant 1$, A_{p_α} has size n for all α, and that $A_{p_\alpha} \cup B_{p_\alpha} < A_{p_\beta} \cup B_{p_\beta}$ for $\alpha < \beta$. For $\alpha < \omega_1$, let b_α be a fixed upper bound of B_{p_α} and let $\{a_\alpha^1, \dots, a_\alpha^n\}$ be the increasing enumeration of A_{p_α}. Define

$$X_1 = \{ a \in D \colon \{ \alpha < \omega_1 \colon a \leqslant a_\alpha^1 \} \text{ is uncountable} \}.$$

Then by the choice of E, X_1 is an uncountable subset of D. So there is a countable set $C_1 \subseteq X_1$ with no upper bound in D. Pick a $c_1 \in C_1$ such that the sets

$$Z_1 = \{ \alpha < \omega_1 \colon c_1 \nleqslant_D b_\alpha \} \quad \text{and} \quad Y_1 = \{ \alpha < \omega_1 \colon c_1 \leqslant_D a_\alpha^1 \}$$

are both uncountable. Note that for each $\alpha \in Y_1$ and $\beta \in Z_1$ with $\alpha < \beta$, $a_\alpha^1 \nleqslant_D b$ for all $b \in B_{p_\beta}$. Define now

$$X_2 = \{ a \in D \colon \{ \alpha \in Y_1 \colon a \leqslant a_\alpha^2 \} \text{ is uncountable} \}.$$

By the definition of E, X_2 is an uncountable subset of D so there is a countable set $C_2 \subseteq X_2$ with no upper bound in D. Pick $c_2 \in C_2$ such that

$$Z_2 = \{ \alpha \in Z_1 \colon c_2 \nleqslant_D b_\alpha \} \quad \text{and} \quad Y_2 = \{ \alpha \in Y_1 \colon c_1 \leqslant a_\alpha^2 \}$$

are both uncountable. Iterating this procedure, we obtain sequences $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$ and $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n$ of uncountable subsets of ω_1 such that for all $\alpha \in Y_i$ and $\beta \in Z_i$ with $\alpha < \beta$, $a_\alpha^j \nleqslant_D b$ for all $1 \leqslant j \leqslant i$ and $b \in B_{p_\beta}$. Thus if $\alpha \in Y_n$ and $\beta \in Z_n$ are such that $\alpha < \beta$, then p_α and p_β are compatible in $\mathscr{P}_1$. This proves the claim.

Pick a filter $\mathscr{G}_1 \subseteq \mathscr{P}_1$ which intersects each of the dense sets $\{ p \in \mathscr{P}_1 \colon b \in B_p$ and $\max A_p \geqslant b \}$ where $b \in E$. Let $A = \bigcup \{ A_p \colon p \in \mathscr{G}_1 \}$. Then A is an uncountable subset of D such that each infinite subset of A is unbounded in D. This shows that $D \equiv [\omega_1]^{<\omega}$.

Case II. An elementary chain of submodels as in Case I does not exist. So there must be a limit nonzero ordinal $\delta < \omega_1$ and a strictly increasing continuous sequence $\langle N_\xi \colon \xi \leqslant \delta \rangle$ of countable elementary submodels of $H_{\aleph_3}$ containing D such that the following set is cofinal in D.

$$D' = \{ d \in D \colon \text{there is a } \xi < \delta \text{ such that } c \nleqslant_D d \text{ for all } c \in N_{\xi+1} \setminus N_\xi \}.$$

Assume δ is a minimal limit countable ordinal with this property. Then for every $b \in D$ and $\gamma < \delta$ there must be $d \geqslant_D b$ in D' and $\xi \in (\gamma, \delta)$ such that $c \nleqslant_D d$ for all $c \in N_{\xi+1} \setminus N_\xi$.

Define $\mathscr{M}$ to be the set of all countable $M \prec H_{\aleph_2}$ containing D such that for some limit nonzero $\delta_M < \omega_1$ and a strictly increasing continuous sequence $\langle M_\xi \colon \xi < \delta_M \rangle$ of countable elementary submodels of $H_{\aleph_2}$, the following conditions are satisfied:

(a) $M = \bigcup_{\xi < \delta_M} M_\xi$;

(b) $D_M = \{ d \in D \colon$ there is a $\xi < \delta_M$ such that $c \nleqslant_D d$ for all $c \in M_{\xi+1} \setminus M_\xi \}$ is cofinal in D;

(c) for all $b \in D$ and $\gamma \in \delta_M$ there exists $d \geqslant b$ in D_M and $\xi \in (\gamma, \delta_M)$ such that $c \nleqslant_D d$ for all $c \in M_{\xi+1} \setminus M_\xi$.

Then $\mathscr{M}$ is a stationary subset of $[H_{\aleph_2}]^{\aleph_0}$ since clearly $\mathscr{M} \in N_\delta$ and $N_\delta \cap H_{\aleph_2} \in \mathscr{M}$.

Let $\mathscr{P}_2$ be the set of all pairs $p = \langle A_p, B_p \rangle$, where A_p and B_p are finite subsets of D. The ordering on $\mathscr{P}_2$ is defined by $q \leqslant p$ iff $A_q \supseteq A_p$, $B_q \supseteq B_p$, and $a \nleqslant_D b$ for all $a \in A_q \setminus A_p$ and $b \in B_p$. Working as in the Case I, the following claim finishes our discussion of Case II and also the proof of Theorem 7.

CLAIM 2. $\mathscr{P}_2$ is a ccc poset.

PROOF. Let $\langle p_\alpha \colon \alpha < \omega_1 \rangle$ be a given sequence of elements of $\mathscr{P}_2$. Again we may assume that $A_{p_\alpha} \cup B_{p_\alpha} < A_{p_\beta} \cup B_{p_\beta}$ for $\alpha < \beta$. Since $\mathscr{M}$ is a stationary subset of $[H_{\aleph_2}]^{\aleph_0}$, there is an $M \in \mathscr{M}$ such that $\langle p_\alpha \colon \alpha < \omega_1 \rangle \in M$. Let $\langle M_\xi \colon \xi < \delta_M \rangle$ be a decomposition of M which satisfies (a)–(c). Pick a $\gamma \in \delta_M$ such that $\langle p_\alpha \colon \alpha < \omega_1 \rangle \in M_\gamma$. Fix a $\beta < \omega_1$ such that $B_{p_\beta} \cap M = \varnothing$. By (c) we can find an upper bound d of B_{p_β} in D_M such that for some $\xi \in (\gamma, \delta_M)$, $c \nleqslant_D d$ for all $c \in M_{\xi+1} \setminus M_\xi$. Note that $\langle A_{p_\alpha} \colon \alpha < \omega_1 \rangle \in M_{\xi+1}$, so there is an α in $M_{\xi+1}$ such that $A_{p_\alpha} \cap M_\xi = \varnothing$. By the choice of d and ξ and the fact that $A_{p_\alpha} \subseteq M_{\xi+1} \setminus M_\xi$, we have that $a \nleqslant_D b$ for all $a \in A_{p_\alpha}$ and $b \in B_{p_\beta}$. So p_α and p_β are compatible in $\mathscr{P}_2$. This completes the proof.

COROLLARY 8 ($\mathrm{MA}_{\aleph_1}$). *Let P be an uncountable partially ordered set with no uncountable antichain. Then P contains an uncountable set X such that every countable subset of X has an upper bound in P.*

PROOF. Since P is in particular a ccc poset it contains an uncountable directed subset D. Clearly $D \ngeqslant [\omega_1]^{<\omega}$, so by Theorem 7, there must be an uncountable $X \subseteq D$ such that every countable subset of X has an upper bound in D.

Corollary 8 is a result announced in [**18**] and it is connected with an unpublished problem of Galvin. Let us note that K. Devlin and J. Steprāns [**2**, §5] have previously deduced the conclusion of Theorem 7 from a much stronger forcing axiom PFA. Theorem 7 is in a sense optimal since MA does not imply $\mathscr{D}_{\aleph_1} = \{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}\}$. This can be proved using methods of [**1**].

4. Five cofinal types. In this section we prove the main result of this paper which says that 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$ are the only cofinal types of directed sets of size $\leqslant \aleph_1$ which can be constructed without additional set-theoretic assumptions. This will be done using an iterated forcing construction and we assume the reader is familiar with some basic facts about iterated forcing.

THEOREM 9. *If* ZF *is consistent, then so is* ZFC *plus* MA *plus*

$$\mathscr{D}_{\aleph_1} = \{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}\}.$$

A partially ordered set $\mathscr{P}$ is *proper* [**14**] iff for all large enough regular cardinals θ and all countable $N \prec H_\theta$ which contain $\mathscr{P}$, every condition from $\mathscr{P} \cap N$ can be extended to a condition q which forces that $\dot{G}_{\mathscr{P}} \cap N$ intersect each dense subset of $\mathscr{P}$ which is a member of N. Such a condition q is called an $(N, \mathscr{P})$-generic condition. A basic result of Shelah [**14**] says that countable support iterations preserve properness. This result will essentially be reproduced at the end of this section. Note that any proper poset preserves ω_1.

Let D be a fixed directed set of size $\aleph_1$ such that $\omega \times \omega_1 \not\geqslant D$. We begin the proof of Theorem 9 with a description of our basic poset $\mathscr{P} = \mathscr{P}_D$ which forces $D \equiv [\omega_1]^{<\omega}$. Note that by Proposition 3 this will be a right step toward the model of Theorem 9. We say that a subset E of D is ω-*bounded* in D if every countable subset of E is bounded in D. Clearly $\omega_1 \times \omega \not\geqslant D$ is equivalent to the fact that D is not a countable union of its ω-bounded subsets. As usual we assume that the domain of D is equal to ω_1 and that $\alpha <_D \beta$ implies $\alpha < \beta$.

If N is a countable elementary submodel of $H_{\aleph_2}$, then by $\overline{N}$ we denote the transitive collapse of N. In what follows, we shall add $\{D\}$ as a predicate to $H_{\aleph_2}$, so all submodels of $H_{\aleph_2}$ considered in this section are in fact submodels of this expanded structure. Thus an isomorphism between two elementary submodels of $H_{\aleph_2}$ will have to map D into D. Note that if M is a transitive closure of two submodels N and N' of $H_{\aleph_2}$, then the collapsing maps uniquely determine an isomorphism between N and N' (which maps D into D). Let $\mathscr{TM}$ denote the set of all countable transitive sets. For $M \in \mathscr{TM}$ let $\mathscr{N}_M$ be the set of all $N \prec H_{\aleph_2}$ with transitive collapse equal to M.

For $F \in [\omega_1]^{<\omega}$, let $\langle F \rangle$ denote the sequence from $(\omega_1)^{<\omega}$ which enumerates F in increasing order.

Finally, we are ready to define our poset $\mathscr{P} = \mathscr{P}_D$ as the set of all triples $p = \langle A_p, B_p, \mathscr{N}_p \rangle$, where:

(1) $\mathscr{N}_p$ is a finite function with domain a $\in$-chain from $\mathscr{TM}$,

(2) for all $M \in \operatorname{dom} \mathscr{N}_p$, $\mathscr{N}_p(M)$ is a nonempty finite subset of $\mathscr{N}_M$,

(3) for all $M \in M'$ in $\operatorname{dom} \mathscr{N}_p$ and all $N \in \mathscr{N}_p(M)$ there is an $N' \in \mathscr{N}_p(M')$ such that $N \in N'$,

(4) A_p and B_p are finite subsets of ω_1 such that for all $a < b$ in A_p there is an $M \in \operatorname{dom} \mathscr{N}_p$ such that $a \in M$ but $b \notin M$,

(5) if $a \in A_p$ is not in $M \in \operatorname{dom} \mathscr{N}_p$, then a is not a member of any ω-bounded subset of D lying in $\bigcup \mathscr{N}_p(M)$.

For $p, q \in \mathscr{P}$, we let $q \leqslant p$ iff

(6) $\operatorname{dom} \mathscr{N}_q \supseteq \operatorname{dom} \mathscr{N}_p$ and $\mathscr{N}_q(M) \supseteq \mathscr{N}_p(M)$ for all $M \subsetneq \operatorname{dom} \mathscr{N}_p$,

(7) $A_q \supseteq A_p$, $B_q \supseteq B_p$, and $a \nleqslant_D b$ for all $a \in A_q \setminus A_p$ and $b \in B_p$.

LEMMA 4. *$\mathscr{P}$ is a proper poset.*

PROOF. Let θ be a large enough regular cardinal and let $N_\theta \prec H_\theta$ be countable such that $p, \mathscr{P}, D \in N_\theta$. Define

$$q = \left\langle A_p, B_p, \mathscr{N}_p \cup \left\{ \left\langle \overline{N_\theta \cap H_{\aleph_2}}, \{ N_\theta \cap H_{\aleph_2} \} \right\rangle \right\} \right\rangle.$$

Then $q \in \mathscr{P}$ and $q \leqslant p$. We shall prove that q is an $(N_\theta, \mathscr{P})$-generic condition. So let $\mathscr{D} \in N_\theta$ be a dense open subset of $\mathscr{P}$ and let $r \leqslant q$. We have to show that r is compatible with a member of $\mathscr{D} \cap N_\theta$. Clearly, we may assume $r \in \mathscr{D}$.

Define

$$A_{\bar{r}} = A_r \cap N_\theta, \qquad B_{\bar{r}} = B_r \cap N_\theta$$

and

$$\mathscr{M}_{\bar{r}} = \left\{ M \in \operatorname{dom} \mathscr{N}_r \colon M \in \overline{N_\theta \cap H_{\aleph_2}} \right\}.$$

Then $A_{\bar{r}}$, $B_{\bar{r}}$ and $\mathscr{M}_{\bar{r}}$ are elements of N_θ. Let $N_0, \ldots, N_k$ be a list of all members of $\mathscr{N}_r(\overline{N_\theta \cap H_{\aleph_2}})$ with $N_0 = N_\theta \cap H_{\aleph_2}$. For $i \leqslant k$, let

$$\pi_i \colon \langle N_i, \in, \{D\} \rangle \to \langle N_0, \in, \{D\} \rangle$$

be the induced isomorphism. Note that by (3), for all $M \in \mathscr{M}_{\bar{r}}$ and $N \in \mathscr{N}_r(M)$ the set $I(N) = \{ i \leqslant k \colon N \in N_i \}$ is nonempty. For $M \in \mathscr{M}_{\bar{r}}$, we define

$$\mathscr{N}_{\bar{r}}(M) = \{ \pi_i(N) \colon N \in \mathscr{N}_r(M) \text{ and } i \in I(N) \}.$$

Then it is easily checked that $\bar{r} = \langle A_{\bar{r}}, B_{\bar{r}}, \mathscr{N}_{\bar{r}} \rangle$ is a member of $\mathscr{P} \cap N_\theta$ and that $\bar{r} \leqslant p$.

Let $A = A_r \setminus A_{\bar{r}}$. We may assume that A is nonempty since otherwise we are easily done. Let n be the size of A and define

$$X = \left\{ x \in (\omega_1)^n \colon A_{\bar{r}} < x = \langle A_s \setminus A_{\bar{r}} \rangle \text{ for some } s \leqslant \bar{r} \text{ in } \mathscr{D} \right\}.$$

Define inductively $X_0, \ldots, X_n$ such that $X_0 = X$ and

$$X_i = \left\{ x \in (\omega_1)^{n-i} \colon \{ \alpha < \omega_1 \colon x^\frown\langle \alpha \rangle \in X_{i-1} \} \text{ is not } \omega\text{-bounded in } D \right\}$$

for $0 < i \leqslant n$. Clearly, $X_i \in N_\theta \cap H$ for all $i \leqslant n$.

CLAIM 3. $\langle\ \rangle \in X_n$.

PROOF. Let $\{a_0, \ldots, a_{n-1}\}$ be the increasing enumeration of A and let $x = \langle a_0, \ldots, a_{n-1} \rangle$. It suffices to show, by induction on $i \leqslant n$, that $x \upharpoonright (n-i) \in X_i$ for all $i \leqslant n$. Since r satisfies (3) we can pick a chain $N^0 \in \cdots \in N^{n-1}$ in $\bigcup \operatorname{range} \mathscr{N}_r$ such that

(a) $N^0 = N_\theta \cap H_{\aleph_2}$,

(b) $a_i \notin N^i$ for all $i < n$,

(c) $a_i \in N^{i+1}$ for all $i < n - 1$.

The induction step from $i - 1$ to i follows from the facts that $X_i, X_{i-1} \in N^{i-1}$, $x \upharpoonright (n-i)^\frown \langle a_{n-i} \rangle \in X_{i-1}$ and a_{n-i} is not in any ω-bounded subset of D lying in N^{i-1}. This finishes the proof.

Since $\langle\ \rangle \in X_n$, the set $E_0 = \{e \in D\colon \langle e\rangle \in X_{n-1}\}$ is not ω-bounded in D. Clearly $E_0 \in N_\theta$, so there must be an $e_0 \in E_0 \cap N_\theta$ such that $e_0 \nleq_D b$ for all $b \in B_r$. Since $\langle e_0\rangle \in X_{n-1}$, the set $E_1 = \{e \in D\colon e_0 < e$ and $\langle e_0, e\rangle \in X_{n-2}\}$ is not ω-bounded in D. Since clearly $E_1 \in N_\theta$, there must be an $e_1 \in E_1 \cap N_\theta$ such that $e_1 \nleq_D b$ for all $b \in B_r$. Continuing in this way, we obtain a sequence $e_0 < e_1 < \cdots < e_{n-1}$ in $N_\theta \cap \omega_1$ such that $\langle e_0,\ldots,e_{n-1}\rangle \in X_0 = X$, and such that $e_i \nleq_D b$ for all $i < n$ and $b \in B_r$. Pick an $s \leqslant \bar{r}$ in $\mathscr{D} \cap N_\theta$ such that $A_{\bar{r}} < \{e_0,\ldots,e_{n-1}\} = A_s \setminus A_{\bar{r}}$. Define $\bar{s} = \langle A_{\bar{s}}, B_{\bar{s}}, \mathscr{N}_{\bar{s}}\rangle$ as follows:

(a) $A_{\bar{s}} = A_s \cup A_r$ and $B_{\bar{s}} = B_s \cup B_r$,

(b) $\operatorname{dom} \mathscr{N}_{\bar{s}} = \operatorname{dom} \mathscr{N}_s \cup \operatorname{dom} \mathscr{N}_r$,

(c) $\mathscr{N}_{\bar{s}}(M) = \mathscr{N}_r(M)$ for $M \in \operatorname{dom} \mathscr{N}_r \setminus \operatorname{dom} \mathscr{N}_s$,

(d) $\mathscr{N}_{\bar{s}}(M) = \mathscr{N}_s(M) \cup \{\pi_i^{-1}(N)\colon i \leqslant k$ and $N \in \mathscr{N}_s(M)\}$ for $M \in \operatorname{dom} \mathscr{N}_s$.

CLAIM 4. $\bar{s} \in \mathscr{P}$ and $\bar{s} \leqslant r, s$.

PROOF. Clearly $\bar{s}$ satisfies (1) and (4). (2), (3) and (5) follow from the corresponding conditions of r and s and the choice of the π_i's. The fact that $\bar{s} \leqslant s$ is obvious from the definition of $\bar{s}$ since $A_{\bar{s}} \setminus A_s = A$ is above B_s. The condition (6) for $\bar{s} \leqslant r$ follows from the definitions of $\bar{r}$ and $\bar{s}$ using the isomorphisms π_i's. The condition (7) for $\bar{s} \leqslant r$ follows from the choice of the set $\{e_0,\ldots,e_{n-1}\} = A_s \setminus A_{\bar{r}} = A_{\bar{s}} \setminus A_r$. This completes the proof of Claim 4 and also the proof that q is an $(N_\theta, \mathscr{P})$-generic condition.

LEMMA 5. *$\mathscr{D}_b = \{p \in \mathscr{P}\colon b \in B_p$ and $\max A_p \geqslant b\}$ is a dense open subset of $\mathscr{P}$ for all b in D.*

PROOF. Let $q \in \mathscr{P}$ be given. Pick countable $N \prec H_{\aleph_2}$ such that $b, q \in N$. Since D is not a countable union of its ω-bounded subsets there is an $a \in D \setminus N$ which is not in any ω-bounded subset of D lying in N. Define $A_p = A_q \cup \{a\}$, $B_p = B_q \cup \{b\}$ and $\mathscr{N}_p = \mathscr{N}_q \cup \{\langle \overline{N}, \{N\}\rangle\}$. Then $p = \langle A_p, B_p, \mathscr{N}_p\rangle$ is a member of $\mathscr{D}_b$ which extends q.

Suppose p and q are two conditions from $\mathscr{P}$ such that $A_p = A_q$, $B_p = B_q$ and $\operatorname{dom} \mathscr{N}_p = \operatorname{dom} \mathscr{N}_q$. Define $r = \langle A_r, B_r, \mathscr{N}_r\rangle$ as follows:

(a) $A_r = A_p$ and $B_r = B_p$,

(b) $\operatorname{dom} \mathscr{N}_r = \operatorname{dom} \mathscr{N}_p$, and

(c) $\mathscr{N}_r(M) = \mathscr{N}_p(M) \cup \mathscr{N}_q(M)$ for $M \in \operatorname{dom} \mathscr{N}_r$.

Then it is easily seen that r is the greatest lower bound of p and q in $\mathscr{P}$, so in particular p and q are compatible in $\mathscr{P}$. Hence if CH holds, then $\mathscr{P}$ satisfies the $\aleph_2$-chain condition. Actually, the $\aleph_2$-chain condition of $\mathscr{P}$ is strong enough to be preserved under countable support iterations of any length $\leqslant \omega_2$. This can be proved via standard arguments using certain canonical names for reals which code models from $\operatorname{dom} \mathscr{N}_p$ for $p \in \mathscr{P}$. While this traditional method works, it is far less general and elegant than the following scheme of Shelah [**14**, VIII, §2] which our poset $\mathscr{P}$ satisfies and which will be reproduced here in some detail.

A poset $\mathscr{Q}$ satisfies the $\aleph_2$-isomorphism condition ($\aleph_2$-ic) iff the following holds for all large enough regular cardinals θ where a well-ordering $<$ of H_θ is added as a predicate: If $\alpha < \beta < \omega_2$, if $\alpha \in N_\alpha \prec H_\theta$ and $\beta \in N_\beta \prec H_\theta$ are countable such that

$\mathcal{Q} \in N_\alpha \cap N_\beta$, $N_\alpha \cap \omega_2 \subseteq \beta$, $N_\alpha \cap \alpha = N_\beta \cap \beta$, if $p \in N_\alpha$ and if $\pi\colon N_\alpha \to N_\beta$ is an isomorphism such that $\pi(\alpha) = \pi(\beta)$ and $\pi \upharpoonright N_\alpha \cap N_\beta = \mathrm{id}$, then there is an $(N_\alpha, \mathcal{Q})$-generic condition $q \leqslant p, \pi(p)$ such that

$$q \Vdash \pi''\dot{G}_{\mathcal{Q}} \cap N_\alpha = \dot{G}_{\mathcal{Q}} \cap N_\beta.$$

Roughly speaking, this condition is saying (among other things) that if p and $\pi(p)$ are two conditions with isomorphic countable "histories' which use only ordinals $< \omega_2$, then they are compatible in $\mathcal{Q}$. So if CH holds, then any $\aleph_2$-ic poset satisfies the $\aleph_2$-cc and preserves ω_1. It is also clear that any proper poset of size $\aleph_1$ satisfies $\aleph_2$-ic. Note that the condition q is also $(N_\beta, \mathcal{Q})$-generic, and that q forces π to naturally extend to an isomorphism of $N_\alpha[\dot{G}_{\mathcal{Q}}]$ and $N_\beta[\dot{G}_{\mathcal{Q}}]$.

LEMMA 6. *$\mathcal{P}_D$ satisfies the $\aleph_2$-ic.*

PROOF. Let α, β, N_α, N_β, π and p satisfy the hypothesis of $\aleph_2$-ic. Since N_α and N_β have the same reals and ordinals $< \omega_1$, it follows that

$$A_p = A_{\pi(p)}, \quad B_p = B_{\pi(p)} \quad \text{and} \quad \operatorname{dom} \mathcal{N}_p = \operatorname{dom} \mathcal{N}_{\pi(p)}.$$

Let $r = p \wedge \pi(p)$, and define $q \in \mathcal{P}$ by

(a) $A_q = A_r$, $B_q = B_r$, and

(b) $\mathcal{N}_q = \mathcal{N}_r \cup \{\langle \overline{N_\alpha \cap H_{\aleph_2}}, \{N_\alpha \cap H_{\aleph_2}, N_\beta \cap H_{\aleph_2}\}\rangle\}$.

Then q satisfies the conclusion of $\aleph_2$-ic. The proof that q is an $(N_\alpha, \mathcal{P})$-generic condition is almost the same as the proof of Lemma 4. The fact that

$$q \Vdash \pi''\dot{G}_{\mathcal{P}} \cap N_\alpha = \dot{G}_{\mathcal{P}} \cap N_\beta$$

follows easily from the compatibility conditions of the poset $\mathcal{P} = \mathcal{P}_D$. This finishes the proof.

Let $\langle \mathcal{P}_\xi\colon \xi \leqslant \omega_2\rangle$ be a countable support iteration of some posets of the form $\mathcal{P}_D$ and some ccc posets of size $\aleph_1$. Using a standard diagonalization argument [**15**], in order to prove that $\langle \mathcal{P}_\xi\colon \xi \leqslant \omega_2\rangle$ can be chosen in such a way that $\mathcal{P}_{\omega_2}$ forces the conclusion of Theorem 9, it suffices to show that $\mathcal{P}_{\omega_2}$ satisfies the $\aleph_2$-cc. This is a consequence of a general result of Shelah [**14**, VIII, 2.4]. In order to make this paper self-contained we sketch the argument from [**14**].

LEMMA 7. *For all $\xi < \omega_2$, $\mathcal{P}_\xi$ satisfies the $\aleph_2$-ic.*

PROOF. Let $\bar{\xi}$ be a fixed ordinal $< \omega_2$ and let α, β, N_α, N_β, p, $\mathcal{P}_{\bar{\xi}}$ and π satisfy the hypothesis of $\aleph_2$-ic. Note that $N_\alpha \cap \bar{\xi} = N_\beta \cap \bar{\xi}$. By induction on $\xi \leqslant \bar{\xi}$ in N_α, we shall prove the following stronger result:

(i_ξ) *For all $\zeta < \xi$ in N_α, all $p \in N_\alpha \cap \mathcal{P}_\xi$ and all $q_\zeta \in \mathcal{P}_\zeta$ with $q_\zeta \leqslant p \upharpoonright \zeta$, $\pi(p) \upharpoonright \zeta$, if q_ζ is $(N_\alpha, \mathcal{P}_\zeta)$-generic and $q_\zeta \Vdash \pi''\dot{G}_\zeta \cap N_\alpha = \dot{G}_\zeta \cap N_\beta$, then there is an $(N_\alpha, \mathcal{P}_\xi)$-generic $q_\xi \leqslant p \upharpoonright \xi$, $\pi(p) \upharpoonright \xi$ such that $q_\xi \upharpoonright \zeta = q_\zeta$ and $q_\xi \Vdash \pi''\dot{G}_\xi \cap N_\alpha = \dot{G}_\xi \cap N_\beta$.*

The case $\xi = \zeta + 1$ is straightforward, so we assume ξ is a limit ordinal. Pick an increasing sequence $\zeta = \xi_0 < \cdots < \xi_n < \cdots$ in $\xi \cap N_\alpha$ cofinal with $\xi \cap N_\alpha$. Let $\langle \mathcal{D}_n\colon n < \omega\rangle$ be a list of all dense open subsets of $\mathcal{P}_\xi$ which are members of N_α. Let

$q^0 = q_\xi$, and let $\mathscr{A}_0$ be a maximal antichain in $\{r \restriction \xi_1 : r \in \mathscr{D}_0 \text{ and } r \leqslant p\}$ such that $\mathscr{A}_0 \in N_\alpha$. For $r \in \mathscr{A}_0$, we let r^* denote the $<$-minimal element of $\mathscr{D}_0$ such that $r^* \leqslant p$ and $r^* \restriction \xi_1 = r$. For each $r \in \mathscr{A}_0 \cap N_\alpha$, we fix $q^1(r) \leqslant r^* \restriction \xi_1, \pi(r^*) \restriction \xi_1$ which end-extends q^0 and satisfies (i_{ξ_1}). Define $q^1 \in \mathscr{P}_{\xi_1}$ to be equal to q^0 on ξ_0 and to $q^1(r)$ on $[\xi_0, \xi_1)$ if $r \in \mathscr{A}_0 \cap N_\alpha \cap \dot{G}_{\xi_0}$. Now for each $r \in \mathscr{A}_0 \cap N_\alpha$ fix, in N_α, a maximal antichain $\mathscr{A}_1^r$ in $\{s \restriction \xi_2 : s \in \mathscr{D}_1 \text{ and } s \leqslant r^*\}$. For $s \in \mathscr{A}_1^r$, let s^* be the $<$-minimal element of $\mathscr{D}_1$ such that $s^* \leqslant r^*$ and $s^* \restriction \xi_2 = s$. For $r \in \mathscr{A}_0 \cap N_\alpha$ and $s \in \mathscr{A}_1^r \cap N_\alpha$, fix $q^2(s) \leqslant s^* \restriction \xi_2, \pi(s^*) \restriction \xi_2$ which end-extends $q^1(r)$ and satisfies (i_{ξ_2}). Define $q^2 \in \mathscr{P}_{\xi_2}$ to be equal to q^1 on ξ_1 and to $q^2(s)$ on $[\xi_1, \xi_2)$ if s is in $\mathscr{A}_1^r \cap N_\alpha \cap \dot{G}_{\xi_1}$ for some $r \in \mathscr{A}_0 \cap N_\alpha$. Iterating this procedure, we obtain an end-extending sequence $q^n \in \mathscr{P}_{\xi_n}$ $(n < \omega)$ whose union q_ξ satisfies the conclusion of (i_ξ). This completes the proof of Lemma 7 and also the proof of Theorem 9.

The proof of Theorem 9 can also be given via an axiomatic approach [**2**, **14**], but this is only a matter of taste since the proof remains essentially the same. However, if we are willing to use an inaccessible cardinal, then the poset $\mathscr{P}_D$ can be made simpler and there is no need in proving any chain condition for $\mathscr{P}_D$ at all. Namely, in this case, as a side condition in $p \in \mathscr{P}_D$ we use just an $\in$-chain of submodels of $H_{\aleph_2}$. Some information concerning this approach can be found in [**18**].

In [**17**] we have stated the following partition property of ω_1 as a strengthening of a similar partition relation considered in the same paper:

(∗) *For every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$ or else there exist $\langle A_n : n < \omega\rangle$ and $\langle \mathscr{B}_n : n < \omega\rangle$ such that*

(i) *$\omega_1 \setminus \bigcup_n A_n$ is countable*;

(ii) *$\mathscr{B}_n$ is a family of $\aleph_1$ disjoint finite subsets of ω_1*;

(iii) *$(\{\alpha\} \otimes F) \cap K_1 \neq \varnothing$ for all $\alpha \in A_n$ and $F \in \mathscr{B}_n$ with $\alpha < \min F$.*

It should be clear that the poset $\mathscr{P}_D$ can easily be modified so as to give a poset for (∗), so the model of Theorem 9 can also satisfy (∗). In [**17**] we have mentioned that it is impossible to strengthen (∗) by demanding ω_1 to be a countable union of 0-homogeneous sets if there are no A_n's and $\mathscr{B}_n$'s satisfying (i)–(iii). Namely, let

$$T = \{s \in (2)^{<\omega_1} : s^{-1}(1) \text{ is finite}\}$$

and define $[T]^2 = K_0 \cup K_1$ by: $\{s, t\} \in K_1$ iff $s \subset t$.

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